

# A NON EXPLICIT COUNTEREXAMPLE TO A PROBLEM OF QUASI-NORMALITY

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ABSTRACT. In 1986, S.Y. Li and H. Xie proved the following theorem: *Let  $k \geq 2$  and let  $\mathcal{F}$  be a family of functions meromorphic in some domain  $D$ , all of whose zeros are of multiplicity at least  $k$ . Then  $\mathcal{F}$  is normal if and only if the family  $\mathcal{F} = \left\{ \frac{f^{(k)}}{1+(f)^{k+1}} : f \in \mathcal{F} \right\}$  is locally uniformly bounded in  $D$ .*

Here we give, in the case  $k = 2$ , a counterexample to show that if the condition on the multiplicities of the zeros is omitted, then the local uniform boundedness of  $\mathcal{F}_2$  does not imply even quasi-normality. In addition, we give a simpler proof for the Li-Xie theorem that does not use Nevanlinna's Theory which was used in the original proof.

## 1. INTRODUCTION

Marty's Theorem characterizes normality by using the first derivative and it has an obvious geometrical meaning.

H.L. Royden, [3], extended one direction of Marty's Theorem and proved

**Theorem 1.** *Let  $\mathcal{F}$  be a family of meromorphic functions in  $D$ , with the property that for each compact set  $K \subset D$ , there is a positive increasing function  $h_K$  such that*

$$(1) \quad |f'(z)| \leq h_K(|f(z)|)$$

*for all  $f \in \mathcal{F}$  and  $z \in K$ . Then  $\mathcal{F}$  is normal in  $D$ .*

This result was extended further in various directions. In [1], (1) is limited to only 5 values. In [4, Thm.2],  $h_K$  is replaced by a nonnegative function that needs to be bounded in a neighborhood of some  $x_0$ ,  $0 \leq x_0 < \infty$ . Then, in [7] it was shown that it is enough that  $h_K$  be finite only in a single point  $x_0$ ,  $x_0 > 0 < \infty$ . Moreover, in [4, Thm.3], this result is extended further to higher derivatives, i.e., (1) is replaced by  $|f^{(\ell)}(z)| \leq h_K(|f(z)|)$ ,  $f \in \mathcal{F}$ ,  $z \in K$ , where  $\ell \geq 2$  and the members of  $\mathcal{F}$  have zeros

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of multiplicity  $\geq l$ . The following generalization of Marty's Theorem also deals with higher derivatives.

**Theorem 2.** [2] *Let  $\mathcal{F}$  be a family of functions meromorphic on  $D$  such that each  $f \in \mathcal{F}$  has zeros only of multiplicity  $\geq k$ . Then  $\mathcal{F}$  is normal in  $D$  if and only if the family*

$$(2) \quad \mathcal{F}_k = \left\{ \frac{f^{(k)}}{1 + |f^{k+1}|} : f \in \mathcal{F} \right\} \quad \text{is locally uniformly bounded in } D.$$

The direction  $(\Rightarrow)$  in Theorem 2 is true even without the assumption that the zeros of each  $f \in \mathcal{F}$  are of multiplicity at least  $k$ . In Section 2, we give a simpler proof for Theorem 2, without using Nevanlinna's Theory. The condition on the multiplicities of  $f \in \mathcal{F}$  is essential in the direction  $(\Leftarrow)$ . Indeed, let  $\hat{\mathcal{F}}_k$  be the family of all polynomials of degree at most  $k-1$  in some domain  $D \subset \mathbb{C}$ . Then  $\frac{f^{(k)}}{1+|f|^{k+1}} = 0$  for each  $f \in \hat{\mathcal{F}}_k$ , but  $\hat{\mathcal{F}}_k$  is not normal in  $D$ . However,  $\hat{\mathcal{F}}_k$  is a quasi-normal family in  $D$  (of order  $k-1$ ). The question that naturally arises is whether the condition (2) implies quasi-normality.

The conjecture that (2) implies quasi-normality (without the assumption on the multiplicities of the zeros) gets support also from another direction.

First let us set some notation. For  $z_0 \in \mathbb{C}$  and  $r > 0$ ,  $\Delta(z_0, r) = \{z : |z - z_0| < r\}$ . We write  $f_n \xrightarrow{\chi} f$  on  $D$  to indicate that the sequence  $\{f_n\}$  converges to  $f$  in the spherical metric uniformly on compact subsets of  $D$  and  $f_n \Rightarrow f$  on  $D$  if the convergence is in the Euclidean metric.

Let us recall the well-known result of L. Zalcman.

**Lemma 1** (Zalcman's Lemma). [6] *A family  $\mathcal{F}$  of functions meromorphic in some domain  $D$  is not normal at  $z_0 \in D$  if and only if there exist points  $z_n$  in  $D$ ,  $z_n \rightarrow z_0$ ; numbers  $\rho_n \rightarrow 0^+$ , and functions  $f_n \in \mathcal{F}$  such that*

$$(3) \quad f_n(z_n + \rho_n \zeta) \xrightarrow{\chi} g(\zeta) \quad \text{in } \mathbb{C},$$

where  $g$  is a nonconstant meromorphic function in  $\mathbb{C}$ .

Now, suppose that  $g$  is a limit function from (3), and we have some  $C > 0$  and  $r > 0$  such that

$$(4) \quad \frac{|f_n^{(k)}(z)|}{1 + |f_n(z)|^{k+1}} \leq C \quad \text{for every } z \in \Delta(z_0, r) \quad \text{and } n \in \mathbb{N}.$$

Let us denote the poles of  $g$  (if any) by  $P_g$ . Then

$$(5) \quad f_n(z_n + \rho_n \zeta) \Rightarrow g(\zeta) \quad \text{on} \quad \mathbb{C} \setminus P_g.$$

(Here we substitute “ $\overset{X}{\Rightarrow}$ ” by “ $\Rightarrow$ ” since in every compact subset of  $\mathbb{C} \setminus P_g$ ,  $f_n(z_n + \rho_n \zeta)$  is holomorphic for large enough  $n$ ).

Differentiating (5)  $k$  times given

$$\rho_n^k f_n^{(k)}(z_n + \rho_n \zeta) \Rightarrow g^{(k)}(\zeta) \quad \text{in} \quad \mathbb{C} \setminus P_g.$$

But then by (3) and (4), we get that  $g^{(k)} \equiv 0$  in  $\mathbb{C} \setminus P_g$  and so  $g^{(k)} \equiv 0$  in  $\mathbb{C}$ . This implies that  $g$  is a polynomial of degree at most  $k - 1$ . Hence, we get that the collection of all limit functions obtained by (3) is a quasi-normal family.

However, it turns out that without the condition on the multiplicities of the zeros, the family  $\mathcal{F}$  of Theorem 2 is not quasi-normal.

We suffice to construct a detailed counterexample for the case  $k = 2$ . This is the content of Section 3.

## 2. PROOF OF THEOREM 2

Assume first that  $\mathcal{F}$  is locally uniformly bounded in  $D$ , and suppose by negation that  $\mathcal{F}_k$  is not normal at some  $z_0 \in D$ . Then similarly to (3) we get the existence of  $f_n, z_n, \rho_n$  and  $g$  such that  $f_n(z_n + \rho_n \zeta) \overset{X}{\Rightarrow} g(\zeta)$  in  $\mathbb{C}$ . With the same reasoning, we deduce that  $g$  is a polynomial of degree at most  $k - 1$ . But now according to the condition on the multiplicities of the zeros of each  $f_n$ , we get that the zeros of  $g$  also must be of multiplicity at least  $k$ . This implies that  $g$  has no zeros and thus  $g$  is a constant function, a contradiction.

For the proof of the opposite direction, we need the following lemma.

**Lemma 2.** *Let  $\{f_n\}_{n=1}^\infty$  be a sequence of meromorphic functions in a domain  $D$ , satisfying  $f_n \overset{X}{\Rightarrow} \infty$  in  $D$ . Then for every  $\ell \in \mathbb{N}$ ,  $\frac{f_n^{(\ell)}}{f_n^{\ell+1}} \Rightarrow 0$  in  $D$ .*

*Proof.* We apply induction. Since  $\frac{1}{f_n(z)} \Rightarrow 0$  in  $D$ , we can differentiate it and obtain that  $\frac{f'_n(z)}{f_n^2(z)} \Rightarrow 0$  in  $D$ , and this proves the case  $\ell = 1$ .

Assume that the lemma holds for  $m \leq \ell$ . We prove it now for the case  $m = \ell + 1$ . We have  $\frac{f_n^{(\ell)}}{f_n^{\ell+1}}(z) \Rightarrow 0$  in  $D$ , and hence, since  $f_n(z) \Rightarrow \infty$  in  $D$ , also  $\frac{f_n^{(\ell)}(z)}{f_n(z)^{\ell+2}} \Rightarrow 0$  in  $D$ . Differentiating the last convergence gives

$$\frac{f_n^{(\ell+1)}(z)}{f_n^{\ell+2}} - (\ell + 2) \frac{f_n' f_n^{(\ell)}}{f_n^2 f_n^{\ell+1}}(z) \Rightarrow 0 \quad \text{in } D.$$

The induction assumption for  $m = 1$  and  $m = \ell$  implies that the right term in the left hand above converges uniformly to 0 on compacta of  $D$ , and thus also  $\frac{f_n^{(\ell+1)}}{f_n^{\ell+2}}(z) \Rightarrow 0$  in  $D$ , as required.

Let us prove now the opposite direction of Theorem 2. Assume that  $\mathcal{F}$  is normal in  $D$ , and suppose by negation that  $\mathcal{F}_k$  is not locally uniformly bounded in any neighborhood of some  $z_0 \in D$ . Thus, there exist functions  $f_n \in \mathcal{F}$ , and points  $z_n \rightarrow z_0$  such that

$$(6) \quad \frac{f_n^{(k)}(z_n)}{1 + |f_n^{k+1}(z_n)|} \xrightarrow{n \rightarrow \infty} \infty.$$

By the normality of  $\mathcal{F}$ ,  $\{f_n\}_{n=1}^\infty$  has a subsequence that, without loss of generality, we also denote by  $\{f_n\}_{n=1}^\infty$ , such that  $f_n \xrightarrow{X} f$  in  $D$ .

We separate now into cases according to the nature of  $f$ .

**Case 1.1**  $f(z_0) \in \mathbb{C}$ .

For small enough  $r > 0$ ,  $f_n^{(k)}(z) \Rightarrow f^{(k)}(z)$  in  $\Delta(z_0, r)$ , and also  $1 + |f_n^{k+1}(z)| \Rightarrow 1 + |f(z)|^{k+1}$  in  $\Delta(z_0, r)$ . Since  $1 + |f_n(z)|^{k+1} \geq 1$ , we get that  $\frac{f_n^{(k)}(z)}{1 + |f_n(z)|^{k+1}} \Rightarrow \frac{f^{(k)}(z)}{1 + |f(z)|^{k+1}}$  in  $\Delta(z_0, r)$ , a contradiction to (6).

**Case 1.2**  $f(z_0) = \infty$ .

Here, for small enough  $r > 0$ ,  $f$  is holomorphic in  $\Delta'(z_0, r)$  and in addition  $|f_n(z)| \geq 2$  and  $|f(z)| \geq 2$  for large enough  $n$ . Thus  $\frac{f_n(z)}{1 + f_n(z)^{k+1}}$  are holomorphic in  $\Delta(z_0, r)$  for large enough  $n$ . We then get by the maximum principle that

$$\frac{f_n^{(k)}(z)}{1 + f_n(z)^{k+1}} \Rightarrow \frac{f^{(k)}(z)}{1 + f(z)^{k+1}} \quad \text{in } \Delta(z_0, r)$$

and then for large enough  $n$ ,

$$\max_{|z-z_0| \leq r/2} \frac{|f_n^{(k)}(z)|}{1 + |f_n(z)|^{k+1}} \leq \max_{|z-z_0| \leq r/2} \frac{|f_n^{(k)}(z)|}{|1 + f_n(z)^{k+1}|} \leq \max_{|z-z_0| \leq r/2} \frac{|f^{(k)}(z)|}{|1 + f(z)^{k+1}|} + 1.$$

The last expression is a positive constant, that does not depend on  $n$  and this is a contradiction to (6).

**Case 2**  $f = \infty$ .

In this case, we get by Lemma 2 that  $\frac{f_n^{(k)}(z)}{f_n(z)^{k+1}} \Rightarrow 0$  in  $D$ , and this is a contradiction to (6).

### 3. CONSTRUCTING THE COUNTEREXAMPLE

We construct a sequence of holomorphic functions  $\{f_n\}_{n=1}^\infty$ , such that for every  $n \geq 1$  and  $z \in \Delta(0, 2)$ ,  $\frac{|f_n''(z)|}{1+|f_n(z)|^3} \leq 1$  and  $\{f_n\}_{n=1}^\infty$  is not quasi-normal in  $\Delta(0, 2)$ .

Let  $g_n(z) = z^n - 1$ ,  $n \geq 1$ . The zeros of  $g_n$  are all simple,  $g_n(z_\ell^{(n)}) = 0$ ,  $0 \leq \ell \leq n-1$ , where  $z_\ell^{(n)}$  is the  $\ell$ -th root of unity of order  $n$ . Define for every  $n \geq 1$ ,  $h_n = g_n e^{p_n}$ , where  $p_n$  is a polynomial to be determined. We have  $h_n' = (g_n' + g_n p_n') e^{p_n}$ , and  $g_n'(z_\ell^{(n)}) \neq 0$ ,  $0 \leq \ell \leq n-1$ . We want that

$$(7) \quad p_n'(z_\ell^{(n)}) = -g_n''(z_\ell^{(n)})/2g_n'(z_\ell^{(n)}), \quad 0 \leq \ell \leq n-1$$

to get that  $h_n''(z_\ell^{(n)}) = 0$ .

We have

$$h_n^{(3)} = e^{p_n} (g_n^{(3)} + 3g_n''p_n' + 3g_n'p_n'' + g_n p_n^{(3)} + 3g_n'p_n'^2 + 3g_n p_n'p_n'' + g_n p_n'^3)$$

We want that

$$(8) \quad p_n''(z_\ell^{(n)}) = -(g_n^{(3)} + 3g_n''p_n' + 3g_n'p_n'^2)/3g_n' \Big|_{z=z_\ell^{(n)}}, \quad 0 \leq \ell \leq n-1$$

to get  $h_n^{(3)}(z_\ell^{(n)}) = 0$ .

Observe that when (7) is satisfied to determine  $p_n'(z_\ell^{(n)})$ , then as in (7), condition (8) is in fact a condition that depends only on the values of  $g_n$  and its derivatives at the points  $z_\ell^{(n)}$ ,  $0 \leq \ell \leq n-1$ .

We have

$$h_n^{(4)} = e^{p_n} (g_n^{(4)} + 4g_n^{(3)}p_n' + 6g_n''p_n'' + 4g_n'p_n^{(3)} + g_n p_n^{(4)} + 6g_n''p_n'^2 + 12g_n'p_n'p_n'' + 3g_n p_n''^2 + 2g_n p_n'p_n^{(3)} + 4g_n'p_n'^3 + 6g_n p_n'^2p_n'' + g_n p_n'^4),$$

we want that

$$(9) \quad p_n^{(3)}(z_\ell^{(n)}) = -(g_n^{(4)} + 4g_n^{(3)}p_n' + 6g_n''p_n'' + 6g_n'p_n'^2 + 12g_n'p_n'p_n'' + 4g_n'p_n'^3)/4g_n' \Big|_{z=z_\ell^{(n)}},$$

$$0 \leq \ell \leq n-1$$

to get  $h_n^{(4)}(z_\ell^{(n)}) = 0$ . Observe that when (7) and (8) are satisfied to determine  $p'_n(z_\ell^{(n)})$  and  $p''_n(z_\ell^{(n)})$ , then also (9) is in fact a condition that depends only on the values of  $g_n$  and its derivatives at the points  $z_\ell^{(n)}$ ,  $0 \leq \ell \leq n-1$ . By the theory of interpolation [5, p. 52], for every  $n \geq 1$  the conditions (7), (8) and (9) can be achieved with a polynomial  $p_n$  of degree at most  $4n-1$ .

Now, by our construction, for every  $n \geq 1$ ,  $h_n''$  has a zero of multiplicity at least 3 at each point  $z_\ell^{(n)}$ ,  $0 \leq \ell \leq n-1$ , and so  $\frac{h_n''}{h_n^3}$  is holomorphic (in fact, entire) in  $\Delta(0, 2)$ . Thus we have  $\max_{z \in \overline{\Delta(0,2)}} |h_n''(z)/h_n^3(z)| = c_n > 0$ .

Define now for every  $n \geq 1$ ,  $f_n := a_n \cdot h_n$ , where  $|a_n|$  is a large enough constant such that  $\left| \frac{c_n}{a_n^2} \right| \leq 1$  and such that every subsequence of  $\{f_n\}_{n=1}^\infty$  is not normal at any point of  $\partial\Delta = \{z : |z| = 1\}$ . In fact, we can take  $|a_n|$  to be so large such that  $f_n \rightarrow \infty$  locally uniformly in  $\mathbb{C} \setminus \partial\Delta$ .

Now, for  $z = z_\ell^{(n)}$ ,  $0 \leq \ell \leq n-1$ ,  $f_n''(z_\ell^{(n)}) = 0$  and thus the left hand side of (2) is zero. If  $z \neq z_\ell^{(n)}$ ,  $z \in \Delta(0, 2)$ , then  $f_n(z) \neq 0$  and

$$\frac{|f_n''(z)|}{1 + |f_n(z)|^3} \leq \frac{|f_n''(z)|}{|f_n(z)|^3} = \frac{1}{|a_n|^2} \frac{|h_n''(z)|}{|h_n(z)|^3} \leq \frac{c_n}{|a_n|^2} \leq 1$$

and (2) is satisfied (uniformly in  $\Delta(0, 2)$ ). This completes the proof that  $\{f_n\}_{n=1}^\infty$  has the desired properties to be a counterexample.

#### 4. SOME REMARKS

**Remark 1.** We have not obtained an explicit formula for  $f_n$ , and this explains the title of this paper.

**Remark 2.** We have shown in fact a stronger counterexample: The condition that  $\left\{ \frac{f''}{f^3} : f \in \mathcal{F} \right\}$  is locally uniformly bounded does not imply quasi-normality of the family  $\mathcal{F}$ .

**Remark 3.** An interesting open problem is to find a differential inequality (maybe of the sort that was mentioned in this paper) that implies quasi-normality and does not imply normality.

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